Time-Decaying Uniform Stresses Inside an Anisotropic Elliptical Inhomogeneity With Nonuniform Interfacial Slip

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This study addresses the problem of an elastically anisotropic elliptical inhomogeneity bonded to an infinite elastically anisotropic matrix through a linear viscous interface. Our results show that uniform, as well as time-decaying stresses, still exist inside the elliptical inhomogeneity when the interface drag parameter, which is varied along the interface, is properly designed, and when the matrix is subjected to remote uniform antiplane shearing. Interestingly, the internal stresses decay with not one but two relaxation times. Some special cases are discussed in detail to demonstrate the obtained solutions. [DOI: 10.1115/1.4000932]

Keywords: anisotropic elasticity, elliptical inhomogeneity, time-decaying uniform stress field, viscous interface, relaxation time

1 Introduction

It was recently observed that uniform stress field still exists inside an elastically isotropic or anisotropic elliptical inhomogeneity bonded to an unbounded matrix through a linear spring-type imperfect interface [1–3]. In the spring-type imperfect interface model, tractions are continuous but displacements are discontinuous across the interface. More precisely, the jumps in displacement components are proportional, in terms of the "spring-factor-type" interface functions (or interface parameters), to the interface traction components (see, for example, Refs. [4–6] for more details). Apparently there is no time effect in the spring-type interface because this kind of interface can only simulate the elastic behavior of a thin interphase layer.

On the other hand, the Nabarro–Herring or Coble creep behavior of the thin interphase layer can be effectively modeled by a linear viscous interface [7,8]. In the viscous interface model the sliding velocity at the interface is proportional, in terms of the interface drag parameter, to the interfacial shear stress. Furthermore the drag parameter can be varied along the interface to reflect the real scenario of the thickening and thinning of the interphase layer [8,9]. The objective of this research is to show that uniform whereas time-decaying stress fields still exist inside an elastically anisotropic elliptical inhomogeneity with a linear viscous interface when the matrix is subjected to remote uniform antiplane shear stresses once the interface drag parameter is properly designed.

2 Basic Formulations

In a fixed rectangular coordinate system x_i $(i=1,2,3)$ let u_i and σ_{ij} be the displacement and stress, respectively. If the material possesses a symmetry plane $x_3=0$, then the equation of equilibrium under antiplane deformation is simply given by

$$\sigma_{31,1}+\sigma_{32,2}=C_{55}u_{,11}+2C_{45}u_{,12}+C_{44}u_{,22}=0 \tag{1}$$

where $u=u_3$, and C_{44}, C_{45}, and C_{55} are the elastic constants. The positive definiteness of the strain energy density will require that

$$C_{44}>0, \quad C_{55}>0, \quad C_{44}C_{55}-C_{45}^2>0 \tag{2}$$

For the special case of an orthotropic material with the orthotropy axes coinciding with the reference axes, one has $C_{45}=0$.

The general solution of Eq. (1) can be expressed in terms of a single analytic function $f(z_p,t)$ as [10]

$$u=\mathrm{Im}\{f(z_p,t)\}, \quad z_p=x_1+px_2 \tag{3}$$

where

$$p=\frac{-C_{45}+i\sqrt{C_{44}C_{55}-C_{45}^2}}{C_{44}} \tag{4}$$

Remark. The appearance of the real time t in the analytic function f comes from the influence of the viscous interface considered below.

The stresses σ_{31} and σ_{32}, and the stress function ϕ are given by [10]

$$\sigma_{31}+p\sigma_{32}=i\mu\ \mathrm{Im}\{p\}\overline{f'(z_p,t)} \tag{5}$$

$$\phi=\mu\ \mathrm{Re}\{f(z_p,t)\} \tag{6}$$

where the prime denotes differentiation with respect to the complex variable z_p, $\mu=\sqrt{C_{44}C_{55}-C_{45}^2}$, and the stresses σ_{31} and σ_{32} are related to the stress function ϕ through

$$\sigma_{31}=-\phi_{,2}, \quad \sigma_{32}=\phi_{,1} \tag{7}$$

Let T be the antiplane surface traction component on a boundary L. If s is the arc-length measured along L such that, when facing the direction of increasing s, the material is on the right-hand side, it can be shown that

$$T=\frac{d\phi}{ds} \tag{8}$$

Consider now the problem of an elliptical inhomogeneity bonded to an unbounded matrix through a linear viscous interface. The linearly elastic materials occupying the inhomogeneity and the matrix are assumed to be homogeneous and anisotropic with associated elastic constants $C_{44}^{(1)},C_{45}^{(1)},C_{55}^{(1)}$ and $C_{44}^{(2)},C_{45}^{(2)},C_{55}^{(2)}$, respectively. We represent the matrix by the domain $S_2:(x_1^2/a^2)+(x_2^2/b^2)\geq 1$ and assume that the inhomogeneity occupies the elliptical region $S_1:(x_1^2/a^2)+(x_2^2/b^2)\leq 1$. The ellipse $L:(x_1^2/a^2)$

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$+(x_2^2/b^2)=1$, whose semi-major and semi-minor axes are a and b, respectively, denote the inhomogeneity-matrix interface. In what follows, the subscripts 1 and 2 will refer to the regions S_1 and S_2, respectively. At infinity, the matrix is subject to remote uniform antiplane shear stresses σ_{31}^∞ and σ_{32}^∞. The slip boundary condition on the viscous interface L can be expressed as [8,9]

$$\begin{aligned}\phi_1 &= \phi_2 \\ -\frac{d\phi_1}{ds} &= T = \beta(x_1,x_2)(\dot{u}_2 - \dot{u}_1)\end{aligned} \quad \text{on } L \tag{9}$$

where an overdot denotes the derivative with respect to the time t, $\beta(x_1,x_2)$ is a non-negative interface drag parameter, whose values depend on the two coordinates x_1 and x_2 of the interface L, and the increasing s is in the counterclockwise direction of the interface.

Now consider the following mapping function:

$$z_1 = m_1(\zeta) = \frac{1}{2}(a - ip_1 b)\zeta + \frac{1}{2}(a + ip_1 b)\zeta^{-1} \tag{10}$$

which can map an elliptical region with a cut in the $z_1(=x_1^{(1)}+ix_2^{(1)}=x_1+p_1x_2)$-plane onto the annulus $\sqrt{|\rho|}\le|\zeta|\le 1$ and $(\rho=(a+ip_1b)/(a-ip_1b))$ in the ζ-plane. Next we consider another mapping function

$$z_2 = m_2(\zeta) = \frac{1}{2}(a - ip_2 b)\zeta + \frac{1}{2}(a + ip_2 b)\zeta^{-1} \tag{11}$$

which can map the outside of an elliptical region in the $z_2(=x_1^{(2)}+ix_2^{(2)}=x_1+p_2x_2)$-plane onto the outside of the unit circle $|\zeta|\ge 1$ in the ζ-plane. For convenience, we write $f_1(z_1,t)=f_1(m_1(\zeta),t)=f_1(\zeta,t)$ and $f_2(z_2,t)=f_2(m_2(\zeta),t)=f_2(\zeta,t)$. In the following we endeavor to derive the expressions of $f_1(\zeta,t)$ and $f_2(\zeta,t)$.

3 The Time-Dependent Uniform Stresses Inside the Elliptical Inhomogeneity

Equation (9) for the boundary conditions on the viscous interface can be expressed in terms of $f_1(\zeta,t)$ and $f_2(\zeta,t)$ as

$$\begin{aligned}&\Gamma f_1(\zeta,t) + \Gamma\overline{f_1(\zeta,t)} = f_2(\zeta,t) + \overline{f_2(\zeta,t)} \\ &\dot{f}_2(\zeta,t) - \overline{\dot{f}_2(\zeta,t)} - \dot{f}_1(\zeta,t) + \overline{\dot{f}_1(\zeta,t)} \\ &\quad = \frac{\mu_1}{\beta(x_1,x_2)b\sqrt{1+b^*\sin^2\theta}}[\zeta f_1'(\zeta,t) - \overline{\zeta f_1'(\zeta,t)}]\end{aligned} \quad \text{on } \zeta = e^{i\theta} \tag{12}$$

where

$$\Gamma = \frac{\mu_1}{\mu_2}, \quad b^* = \frac{a^2 - b^2}{b^2} \tag{13}$$

In this research the interface drag parameter $\beta(x_1,x_2)$ is chosen in such a way that

$$\beta(x_1,x_2) = \frac{\mu_1}{\gamma b\sqrt{1+b^*\sin^2\theta}} \tag{14}$$

where γ is a non-negative constant. It is observed from Eq. (14) that the interface drag parameter is constant along a circular interface $a=b$ (i.e., $b^*=0$). Once the interface drag parameter is chosen by using Eq. (14), the boundary conditions in Eq. (12) simplify to

$$\begin{aligned}&\Gamma f_1(\zeta,t) + \Gamma\overline{f_1(\zeta,t)} = f_2(\zeta,t) + \overline{f_2(\zeta,t)} \\ &\dot{f}_2(\zeta,t) - \overline{\dot{f}_2(\zeta,t)} - \dot{f}_1(\zeta,t) + \overline{\dot{f}_1(\zeta,t)} \\ &\quad = \gamma[\zeta f_1'(\zeta,t) - \overline{\zeta f_1'(\zeta,t)}]\end{aligned} \quad \text{on } \zeta = e^{i\theta} \tag{15}$$

In order to ensure the uniform stresses inside the elliptical inhomogeneity, it is assumed that

$$f_1(\zeta,t) = \frac{k(t)}{2}[(a - ip_1 b)\zeta + (a + ip_1 b)\zeta^{-1}] \tag{16}$$

where $k(t)$ is a time-dependent complex constant to be determined.

It follows from the first condition in Eq. (15) that

$$f_2(\zeta,t) = \left[\frac{\Gamma k(t)}{2}(a + ip_1 b) + \frac{\Gamma\overline{k(t)}}{2}(a + i\bar{p}_1 b) - \bar{\chi}\right]\zeta^{-1} + \chi\zeta, \quad |\zeta| > 1 \tag{17}$$

where the constant χ is related to the remote uniform shearing stresses through the following:

$$\chi = \frac{i(a - ip_2 b)(\sigma_{31}^\infty + \bar{p}_2\sigma_{32}^\infty)}{2\mu_2\,\mathrm{Im}\{p_2\}} \tag{18}$$

It follows from the second condition in Eq. (15) that

$$\dot{f}_2(\zeta,t) = \left[\frac{\dot{k}(t)}{2}(a + ip_1 b) - \frac{\overline{\dot{k}(t)}}{2}(a + i\bar{p}_1 b) - \frac{\gamma k(t)}{2}(a + ip_1 b) - \frac{\gamma\overline{k(t)}}{2}(a + i\bar{p}_1 b)\right]\zeta^{-1}, \quad |\zeta| > 1 \tag{19}$$

The above two expressions of $\dot{f}_2(\zeta,t)$ must be exactly the same. Thus we arrive at the following state-space equation:

$$\begin{bmatrix}(\Gamma-1)(a+ip_1b) & (\Gamma+1)(a+i\bar{p}_1b) \\ (\Gamma+1)(a-ip_1b) & (\Gamma-1)(a-i\bar{p}_1b)\end{bmatrix}\begin{bmatrix}\dot{k}(t) \\ \overline{\dot{k}(t)}\end{bmatrix} = -\gamma\begin{bmatrix}a+ip_1b & a+i\bar{p}_1b \\ a-ip_1b & a-i\bar{p}_1b\end{bmatrix}\begin{bmatrix}k(t) \\ \overline{k(t)}\end{bmatrix} \tag{20}$$

whose solution can be conveniently given by

$$k(t) = \frac{1}{2}\left[k(0) - \frac{\rho}{|\rho|}\frac{a - ip_1 b}{a + i\bar{p}_1 b}\overline{k(0)}\right]\exp(-\lambda_1 t) + \frac{1}{2}\left[k(0) + \frac{\rho}{|\rho|}\frac{a - ip_1 b}{a + i\bar{p}_1 b}\overline{k(0)}\right]\exp(-\lambda_2 t) \tag{21}$$

where

$$\lambda_1 = \frac{\gamma(1-|\rho|)}{\Gamma + 1 - |\rho|(\Gamma - 1)}, \quad \lambda_2 = \frac{\gamma(1+|\rho|)}{\Gamma + 1 + |\rho|(\Gamma - 1)} \tag{22}$$

with $\lambda_2\ge\lambda_1\ge 0$. $\lambda_1=\lambda_2$ only when $\rho=0$ (or equivalently, $p_1=ia/b$).

At the initial moment, the interface is perfect. Thus $k(0)$ can be easily determined as [10]

$$k(0) = \frac{4(\Gamma+1)\chi + 4\bar{\rho}(1-\Gamma)\bar{\chi}}{(a - ip_1 b)[(\Gamma+1)^2 - (\Gamma-1)^2|\rho|^2]} \tag{23}$$

The time-decaying uniform stress field inside the elliptical inhomogeneity can then be determined as

$$\begin{aligned}\sigma_{31} + p_1\sigma_{32} &= \frac{i\mu_1\,\mathrm{Im}\{p_1\}}{2}\left[\left(\overline{k(0)} - \frac{\bar{\rho}}{|\rho|}\frac{a + i\bar{p}_1 b}{a - ip_1 b}k(0)\right)\exp(-\lambda_1 t) + \left(\overline{k(0)} + \frac{\bar{\rho}}{|\rho|}\frac{a + i\bar{p}_1 b}{a - ip_1 b}k(0)\right)\exp(-\lambda_2 t)\right] \\ &\quad \left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \le 1\right)\end{aligned} \tag{24}$$

which clearly demonstrates that the internal stresses decay with two different relaxation times $1/\lambda_1$ and $1/\lambda_2$.

The time-dependent stresses within the matrix can be easily determined by using Eqs. (5) and (17) such that

$$\sigma_{31}+\bar{p}_2\sigma_{32} = -i\mu_2\,\mathrm{Im}\{p_2\} \times\frac{2\chi\zeta^2+2\bar{\chi}-\Gamma k(t)(a+ip_1b)-\Gamma\overline{k(t)}(a+i\bar{p}_1b)}{(a-ip_2b)\zeta^2-(a+ip_2b)},\quad |\zeta|\geq 1 \tag{25}$$

which evolves into the result of an elliptical hole as $t\rightarrow\infty$.

4 Discussions

In the following we will discuss in more details several special cases to demonstrate the obtained solutions.

4.1 Case 1. When $(k(0)/\overline{k(0)})=(\rho/|\rho|)(a-ip_1b)/(a+i\bar{p}_1b)$, the internal stresses decay fast only with the smaller relaxation time $1/\lambda_2$ such that

$$\sigma_{31}+p_1\sigma_{32}=i\mu_1\,\mathrm{Im}\{p_1\}\overline{k(0)}\,\exp(-\lambda_2 t)\quad\left(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\leq 1\right) \tag{26}$$

4.2 Case 2. On the other hand when $(k(0)/\overline{k(0)})=-(\rho/|\rho|)(a-ip_1b)/(a+i\bar{p}_1b)$, the internal stresses decay slowly only with the larger relaxation time $1/\lambda_1$, such that

$$\sigma_{31}+p_1\sigma_{32}=i\mu_1\,\mathrm{Im}\{p_1\}\overline{k(0)}\exp(-\lambda_1 t)\quad\left(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\leq 1\right) \tag{27}$$

4.3 Case 3. When $\rho=0$ (or equivalently $p_1=ia/b$), then we have $\lambda_1=\lambda_2=\gamma/(\Gamma+1)$. Consequently the internal stresses decay with two identical relaxation times $1/\lambda_1=1/\lambda_2=(\Gamma+1)/\gamma$, such that

$$\frac{\sigma_{31}+p_1\sigma_{32}}{\sigma_{31}^\infty+p_2\sigma_{32}^\infty}=\frac{\Gamma(a+i\bar{p}_2b)}{(\Gamma+1)b\,\mathrm{Im}\{p_2\}}\exp\left(-\frac{\gamma}{\Gamma+1}t\right)\quad\left(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\leq 1\right) \tag{28}$$

Furthermore, if $p_2=p_1=ia/b$, we have

$$\frac{\sigma_{31}}{\sigma_{31}^\infty}=\frac{\sigma_{32}}{\sigma_{32}^\infty}=\frac{2\Gamma}{\Gamma+1}\exp\left(-\frac{\gamma}{\Gamma+1}t\right)\quad\left(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\leq 1\right) \tag{29}$$

which is consistent with that obtained by He and Lim [11] for an isotropic circular inhomogeneity bonded to an isotropic matrix ($a=b$ and $p_2=p_1=i$).

4.4 Case 4. When the materials comprising the matrix and inhomogeneity are the same, and their orientations are also the same (i.e., $\Gamma=1$ and $p_1=p_2=p$), the internal stresses decay in the following manner:

$$\begin{aligned}\frac{\sigma_{31}+p\sigma_{32}}{\sigma_{31}^\infty+p\sigma_{32}^\infty} &=\frac{1}{2}\left(1+\frac{\bar{\rho}}{|\rho|}\frac{a+i\bar{p}b}{a-ipb}\frac{\sigma_{31}^\infty+\bar{p}\sigma_{32}^\infty}{\sigma_{31}^\infty+p\sigma_{32}^\infty}\right)\exp\left[-\frac{\gamma(1-|\rho|)}{2}t\right]\\ &+\frac{1}{2}\left(1-\frac{\bar{\rho}}{|\rho|}\frac{a+i\bar{p}b}{a-ipb}\frac{\sigma_{31}^\infty+\bar{p}\sigma_{32}^\infty}{\sigma_{31}^\infty+p\sigma_{32}^\infty}\right)\exp\left[-\frac{\gamma(1+|\rho|)}{2}t\right]\\ &\left(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\leq 1\right)\end{aligned} \tag{30}$$

4.5 Case 5. When both the inhomogeneity and matrix are orthotropic with $C_{45}^{(1)}=C_{45}^{(2)}=0$ and ρ being real, the internal stresses decay in the following manner:

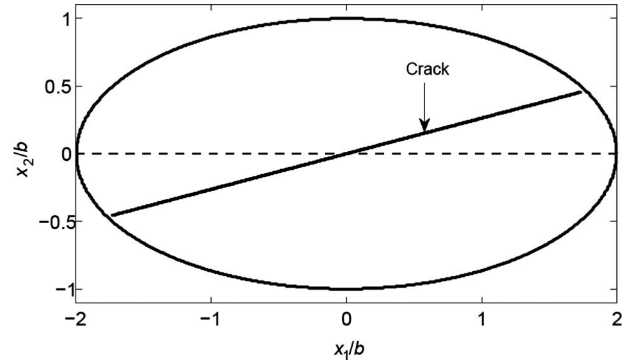

Fig. 1 An anisotropic elliptical inhomogeneity weakened by an internal crack. The geometry of the inhomogeneity is $a/b=2$ and $p_1=1+0.5i$, and the two tips of the crack are located at $[x_1,x_2]=\pm b[1.735,0.4563]$.

$$\begin{aligned}\frac{\sigma_{31}}{\sigma_{31}^\infty}&=\frac{\mathrm{Im}\{p_1\}(a-ip_2b)}{\mathrm{Im}\{p_2\}(a-ip_1b)}\frac{2\Gamma\exp\left[-\frac{\gamma(1-\rho)}{\Gamma+1-\rho(\Gamma-1)}t\right]}{\Gamma+1-\rho(\Gamma-1)},\\ \frac{\sigma_{32}}{\sigma_{32}^\infty}&=\frac{a-ip_2b}{a-ip_1b}\frac{2\Gamma\exp\left[-\frac{\gamma(1+\rho)}{\Gamma+1+\rho(\Gamma-1)}t\right]}{\Gamma+1+\rho(\Gamma-1)}\quad\left(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\leq 1\right)\end{aligned} \tag{31}$$

which clearly indicates that σ_{31} decays only with the relaxation time $(\Gamma+1-\rho(\Gamma-1))/\gamma(1-\rho)$, while σ_{32} decays only with another relaxation time $(\Gamma+1+\rho(\Gamma-1))/\gamma(1+\rho)$.

Furthermore when both the inhomogeneity and matrix are isotropic ($p_1=p_2=i$ and $\rho>0$), the internal stresses decay in the following simple manner:

$$\begin{aligned}\frac{\sigma_{31}}{\sigma_{31}^\infty}&=\frac{2\Gamma\exp(-\lambda_1 t)}{\Gamma+1-\rho(\Gamma-1)},\\ \frac{\sigma_{32}}{\sigma_{32}^\infty}&=\frac{2\Gamma\exp(-\lambda_2 t)}{\Gamma+1+\rho(\Gamma-1)}\quad\left(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\leq 1\right)\end{aligned} \tag{32}$$

which implies that σ_{31} decays slower than σ_{32}.

4.6 Case 6. Now we consider the special case in which the elliptical inhomogeneity is weakened by an internal crack, whose surface is traction-free. Furthermore the two tips of the crack are just located at $z_1=\pm\sqrt{a^2+p_1^2b^2}$. An example of the cracked anisotropic inhomogeneity is illustrated in Fig. 1. When the inhomogeneity is isotropic, the crack tips are just located at the foci of the ellipse $[x_1\;\;x_2]=\pm[\sqrt{a^2-b^2}\;\;0]$. Our main goal below is to detect whether uniform stresses still exist inside the *cracked* elliptical inhomogeneity. Once the interface drag parameter is chosen by using Eq. (14), closed-form solutions of $f_1(\zeta,t)$ and $f_2(\zeta,t)$ can still be easily derived as

$$\begin{aligned}f_1(\zeta,t)&=\frac{2\exp(-\lambda_1 t)(\chi\zeta-|\rho|\bar{\chi}\zeta^{-1})}{\Gamma+1-|\rho|(\Gamma-1)},\quad\sqrt{|\rho|}\leq|\zeta|\leq 1\\ f_2(\zeta,t)&=\bar{\chi}\left[\frac{2\Gamma(1-|\rho|)\exp(-\lambda_1 t)}{\Gamma+1-|\rho|(\Gamma-1)}-1\right]\zeta^{-1}+\chi\zeta,\quad|\zeta|\geq 1\end{aligned} \tag{33}$$

where λ_1 has been defined in Eq. (22). Interestingly the stresses within the cracked inhomogeneity decay only with the single relaxation time $1/\lambda_1$. It can be easily checked that when both the inhomogeneity and matrix are isotropic, $f_1(\zeta,0)$ and $f_2(\zeta,0)$ are just those derived by Wu and Chen [12]. It is further observed that the stresses are still uniform (and then regular) inside the cracked inhomogeneity when the remote shear stresses satisfy the following condition:

$$\frac{\sigma_{31}^{\infty}+p_2\sigma_{32}^{\infty}}{\sigma_{31}^{\infty}+\bar{p}_2\sigma_{32}^{\infty}}=\frac{\rho}{|\rho|}\frac{a-ip_2b}{a+i\bar{p}_2b} \tag{34}$$

When the above condition is met, the internal uniform stress field is explicitly given by

$$\frac{\sigma_{31}+p_1\sigma_{32}}{\sigma_{31}^{\infty}+p_2\sigma_{32}^{\infty}}=\frac{\mathrm{Im}\{p_1\}}{\mathrm{Im}\{p_2\}}\frac{a+i\bar{p}_2b}{a+i\bar{p}_1b}\frac{2\Gamma\exp(-\lambda_1 t)}{\Gamma+1-|\rho|(\Gamma-1)},\quad \sqrt{|\rho|}\le|\zeta|\le 1 \tag{35}$$

Here we have to keep in mind that in the above expression of the internal uniform stress field, the traction is still *zero* along the straight line connecting the two points $z_1=\pm\sqrt{a^2+p_1^2b^2}$.

4.7 Case 7. Finally we consider another special case in which the elliptical inhomogeneity is reinforced by an anticrack (or rigid line). The rigid line can only suffer rigid body displacement. Furthermore the two tips of the anticrack are just located at $z_1=\pm\sqrt{a^2+p_1^2b^2}$. Our main goal below is to detect whether uniform stresses still exist inside the *anticracked* elliptical inhomogeneity. Once the interface drag parameter is chosen by using Eq. (14), closed-form solutions of $f_1(\zeta,t)$ and $f_2(\zeta,t)$ can still be easily derived as

$$f_1(\zeta,t)=\frac{2\exp(-\lambda_2 t)(\chi\zeta+|\rho|\bar{\chi}\zeta^{-1})}{\Gamma+1+|\rho|(\Gamma-1)},\quad \sqrt{|\rho|}\le|\zeta|\le 1$$

$$f_2(\zeta,t)=\bar{\chi}\left[\frac{2\Gamma(1+|\rho|)\exp(-\lambda_2 t)}{\Gamma+1+|\rho|(\Gamma-1)}-1\right]\zeta^{-1}+\chi\zeta,\quad |\zeta|\ge 1 \tag{36}$$

where λ_2 has been defined in Eq. (22). Interestingly the stresses within the anticracked inhomogeneity decay only with the single relaxation time $1/\lambda_2$. It is further observed that the internal stresses are still uniform (and then regular) when the remote shear stresses satisfy the following condition:

$$\frac{\sigma_{31}^{\infty}+p_2\sigma_{32}^{\infty}}{\sigma_{31}^{\infty}+\bar{p}_2\sigma_{32}^{\infty}}=-\frac{\rho}{|\rho|}\frac{a-ip_2b}{a+i\bar{p}_2b} \tag{37}$$

When the above condition is met, the internal uniform stress field is explicitly given by

$$\frac{\sigma_{31}+p_1\sigma_{32}}{\sigma_{31}^{\infty}+p_2\sigma_{32}^{\infty}}=\frac{\mathrm{Im}\{p_1\}}{\mathrm{Im}\{p_2\}}\frac{a+i\bar{p}_2b}{a+i\bar{p}_1b}\frac{2\Gamma\exp(-\lambda_2 t)}{\Gamma+1+|\rho|(\Gamma-1)},\quad \sqrt{|\rho|}\le|\zeta|\le 1 \tag{38}$$

Here we have to keep in mind that in the above expression of the internal uniform stress field, the corresponding displacement is *constant* along the straight line connecting the two points $z_1=\pm\sqrt{a^2+p_1^2b^2}$ (or equivalently, the tangential derivative of the displacement along this straight line is zero).

5 Conclusions

In this research we found that time-decaying uniform stresses still exist inside an anisotropic elliptical inhomogeneity with a linear viscous interface when the interface drag parameter $\beta(x_1,x_2)$ is chosen by using Eq. (14). The choice of the interface drag parameter is independent of the anisotropy of both the elliptical inhomogeneity and the surrounding matrix, and is only reliant on the shape of the ellipse L. The explicit expression of the time-decaying uniform internal stress field with two positive real relaxation times $1/\lambda_1$ and $1/\lambda_2$ was presented in Eq. (24). When only time-independent uniform antiplane eigenstrains ε_{31}^* and ε_{32}^* are imposed on the inhomogeneity, the stress field inside the elliptical inhomogeneity is still uniform, whereas time-decaying once $\beta(x_1,x_2)$ is designed by using Eq. (14). In this case the internal stress field can still be expressed by Eq. (24), but $k(0)$ is now given by

$$k(0)=\frac{4(\Gamma+1)(-b\varepsilon_{32}^*-ia\varepsilon_{31}^*)+4\bar{\rho}(1-\Gamma)(-b\varepsilon_{32}^*+ia\varepsilon_{31}^*)}{(a-ip_1b)[(\Gamma+1)^2-(\Gamma-1)^2|\rho|^2]} \tag{39}$$

which can be proved to recover the result of Shen et al. [13, Eq. (28)] for a composite composed of isotropic constituents ($p_1=p_2=i$). We also showed that the time-decaying uniform stress field still exists inside a cracked or anticracked elliptical inhomogeneity under special loading conditions (see Eqs. (34) and (37)).

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